SOLUTION TO AMM PROBLEM 12480

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Problem 12480. *Proposed by F. Stanescu (Romania)*

Let f and g be two nonnegative functions on $[0,1]$ with $f(0) = g(0) = 1$, and let $h\colon [0,1] \to \mathbb{R}$ be nondecreasing. Suppose that f is convex and g is concave. Prove

$$\int_0^1 g(x)\,\mathrm{d}x \int_0^1 f(x)h(x)\,\mathrm{d}x \geq \int_0^1 f(x)\,\mathrm{d}x \int_0^1 g(x)h(x)\,\mathrm{d}x$$

Solution. *We show a slightly more general statement: let f and g be two nonnegative functions on $[0,1]$ with $f(0) = g(0) = \alpha \geq 0$, and let $h\colon [0,1] \to \mathbb{R}$ be nondecreasing. Suppose that f is convex and g is concave. Then*

$$\int_0^1 g(x)\,\mathrm{d}x \int_0^1 f(x)h(x)\,\mathrm{d}x \geq \int_0^1 f(x)\,\mathrm{d}x \int_0^1 g(x)h(x)\,\mathrm{d}x$$

First, let us consider the case in which $\int_0^1 f(x)\,\mathrm{d}x = 0$. Since f is convex in a compact, it is continuous. Therefore, since f is nonnegative, the condition $\int_0^1 f(x)\,\mathrm{d}x = 0$ implies $f \equiv 0$ in $[0,1]$. The desired relation follows with equality. The same argument applies to g.

So let us assume that $\int_0^1 f(x)\,\mathrm{d}x > 0 < \int_0^1 g(x)\,\mathrm{d}x$. Then, we can divide both sides by $\int_0^1 f(x)\,\mathrm{d}x \cdot \int_0^1 g(x)\,\mathrm{d}x$. We can assume then, without loss of generality, that

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x = 1\,.$$

The problem can be then rewritten as:

$$\int_0^1 h(x)f(x)\,\mathrm{d}x \geq \int_0^1 h(x)g(x)\,\mathrm{d}x$$

or equivalently

$$\int_0^1 \Big(f(x) - g(x)\Big)h(x)\,\mathrm{d}x \geq 0\,.$$

Note that

- $f - g$ is a convex function, since $-g$ is convex;
- $f(0) - g(0) = \alpha - \alpha = 0$, by hypothesis;
- $\int_0^1 (f(x) - g(x))\,\mathrm{d}x = 0$, because we assumed $\int_0^1 f(x)\,\mathrm{d}x = 1 = \int_0^1 g(x)\,\mathrm{d}x$.

Hence, there must be a point $t_0 \in [0, 1)$ such that $f(x) - g(x) \leq 0$ for $x \in (0, t_0)$ and $f(x) - g(x) \geq 0$ for $x \in (t_0, 1)$.

Therefore

$$\begin{aligned}\int_0^1 \Big(f(x) - g(x)\Big) h(x)\,\mathrm{d}x &= \int_0^{t_0} \Big(f(x) - g(x)\Big) h(x)\,\mathrm{d}x + \int_{t_0}^1 \Big(f(x) - g(x)\Big) h(x)\,\mathrm{d}x \\ &\geq h(t_0) \int_0^{t_0} (f(x) - g(x))\,\mathrm{d}x + h(t_0) \int_{t_0}^1 (f(x) - g(x))\,\mathrm{d}x, \\ &= h(t_0) \int_0^1 (f(x) - g(x))\,\mathrm{d}x = 0\end{aligned}$$

where we used the fact that $f(x) - g(x) \leq 0$ in $[0, t_0]$.

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