SOLUTION TO PROBLEM 12407

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Problem 12407

Proposed by an anonymous contributor, New Delhi, India. Let r be a positive real. Evaluate

$$\int_0^\infty \frac{x^{r-1}}{(1+x^2)(1+x^{2r})}\,\mathrm{d}x.$$

Solution. Integration by parts yields

$$\begin{aligned}
&\int_0^\infty \frac{x^{r-1}}{(1+x^2)(1+x^{2r})}\,\mathrm{d}x = \int_0^\infty \frac{1}{(1+x^2)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{r}\arctan(x^r)\right)\mathrm{d}x \\
&= \left(\frac{\arctan x^r}{r(1+x^2)}\right)\Bigg|_0^\infty + \frac{2}{r}\int_0^\infty \arctan(x^r)\frac{x}{(1+x^2)^2}\,\mathrm{d}x \\
&= \frac{2}{r}\int_0^\infty \arctan(x^r)\frac{x}{(1+x^2)^2}\,\mathrm{d}x = \frac{2}{r}\,I(r)
\end{aligned}$$

Using the formula $\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$ and the substitution $t = \frac{1}{x}$, we have

$$\begin{aligned}
I(r) &= \int_0^\infty \arctan(x^r)\frac{x}{(1+x^2)^2}\,\mathrm{d}x = \int_0^\infty \left(\frac{\pi}{2} - \arctan\frac{1}{x^r}\right)\frac{x}{(1+x^2)^2}\,\mathrm{d}x \\
&= \frac{\pi}{2}\int_0^\infty \frac{x}{(1+x^2)^2}\,\mathrm{d}x - \int_0^\infty \arctan\left(\frac{1}{x^r}\right)\frac{x}{(1+x^2)^2}\,\mathrm{d}x \\
&= \frac{\pi}{2}\left(-\frac{1}{2(x^2+1)}\right)\Bigg|_0^\infty - \int_0^\infty \arctan(t^r)\frac{1}{t\left(1+\frac{1}{t^2}\right)^2}\frac{1}{t^2}\,\mathrm{d}t \\
&= \frac{\pi}{4} - \int_0^\infty \arctan(t^r)\frac{t}{(1+t^2)^2}\,\mathrm{d}t \\
&= \frac{\pi}{4} - I(r)
\end{aligned}$$

This implies $I(r) = \frac{\pi}{8}$. Finally,

$$\int_0^\infty \frac{x^{r-1}}{(1+x^2)(1+x^{2r})}\,\mathrm{d}x = \frac{2}{r}\,I(r) = \frac{\pi}{4r}$$