SOLUTION TO PROBLEM 12406

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Problem 12406

Proposed by Raymond Mortini, University of Luxembourg, Esch=sur-Alzette, Luxembourg, and Rudolf Rupp, Nuremberg Institute of Technology, Nuremberg, Germany For fixed $p \in \mathbb{R}$, find all functions $f\colon [0,1] \to \mathbb{R}$ that are continuous at 0 and 1 and satisfy $f(x^2) + 2p \cdot f(x) = (x+p)^2$ for all $x \in [0,1]$.

Solution. First, note that if $p = 0$, then we have $f(x) = x$.

If $p = -\frac{1}{2}$, no such function f satisfying the desired condition exists. Indeed the condition $f(x^2) - f(x) = (x - \frac{1}{2})^2$ yields $0 = \frac{1}{4}$ for $x = 0$, which is false.

Assume $p \notin \{0, -\frac{1}{2}\}$. Then $f(0) = \frac{p^2}{1+2p}$ and $f(1) = \frac{(p+1)^2}{1+2p}$.

Let $r(x)$ be the linear function

$$r(x) := x + \frac{p^2}{1+2p};$$

note $r(0) = f(0)$ and $r(1) = f(1)$. Moreover $f(x) = r(x)$ is a solution of the functional equation. We are going to prove that it is the only one.

Let $h(x) := f(x) - r(x)$. Then h is continuous at 0 and 1, $h(0) = h(1) = 0$ and

$$h(x^2) + 2p \cdot h(x) = 0 \quad \text{for every} \quad x \in [0,1]. \tag{1}$$

If $|p| \geq \frac{1}{2}$ then

$$|h(x)| = \left| \frac{1}{2p}\, h(x^2) \right| = \cdots = \left| \frac{1}{(2p)^n}\, h\left(x^{2^n}\right) \right|; \tag{2}$$

now, $\left|\frac{1}{(2p)^n}\right| \leq 1$ for every n and $\lim_{n\to\infty} x^{2^n} = 0$; since h is continuous at 0, we obtain $|h(x)| = \lim_{n\to\infty} \left| \frac{1}{(2p)^n} h\left(x^{2^n}\right) \right| = 0$.

If $|p| \leq 1/2$ then

$$|h(x)| = |2p \cdot h(x^{1/2})| = \cdots = |(2p)^n \cdot h(x^{1/2^n})|;$$

now, $|(2p)^n| \leq 1$ for every n and $\lim_{n\to\infty} x^{1/2^n} = 1$; since h is continuous at 1, we obtain $|h(x)| = \lim_{n\to\infty} |(2p)^n h(x^{1/2^n})| = 0$.

Thus, $h(x) \equiv 0$ and the only possible solution is the linear function $f(x) = r(x)$.