SOLUTION TO PROBLEM 12388

FULVIO GESMUNDO AND TOMMASO MANNELLI MAZZOLI

Problem 12388

Proposed by A.Garcia(France) Let α be a real number. Evaluate

$$I(\alpha) = \int_0^\infty \frac{\ln^2(x)\arctan(x)}{x^2 - 2\cos(\alpha)x + 1}\,\mathrm{d}x$$

Solution.

$$\begin{aligned}
I(\alpha) &= \int_0^1 \frac{\ln^2(x)\arctan(x)}{x^2 - 2\cos(\alpha)x + 1}\,\mathrm{d}x + \int_1^\infty \frac{\ln^2(x)\arctan(x)}{x^2 - 2\cos(\alpha)x + 1}\,\mathrm{d}x \\
&= \int_0^1 \frac{\ln^2(x)\arctan(x)}{x^2 - 2\cos(\alpha)x + 1}\,\mathrm{d}x + \int_0^1 \frac{\ln^2(x)\arctan(\frac{1}{x})}{x^2 - 2\cos(\alpha)x + 1}\,\mathrm{d}x \\
&= \int_0^1 \frac{\ln^2(x)\arctan(x)}{x^2 - 2\cos(\alpha)x + 1}\,\mathrm{d}x + \int_0^1 \frac{\ln^2(x)(\frac{\pi}{2} - \arctan(x))}{x^2 - 2\cos(\alpha)x + 1}\,\mathrm{d}x \\
&= \frac{\pi}{2}\int_0^1 \frac{\ln^2(x)}{x^2 - 2\cos(\alpha)x + 1}\,\mathrm{d}x.
\end{aligned}$$

Using $\cos(\alpha) = \frac{\mathrm{e}^{\mathrm{i}\,\alpha} + \mathrm{e}^{-\mathrm{i}\,\alpha}}{2}$, we obtain

$$x^2 - 2\cos(\alpha)x + 1 = x^2 - x\left(\mathrm{e}^{\mathrm{i}\,\alpha} + \mathrm{e}^{-\mathrm{i}\,\alpha}\right) + 1 = \left(x - \mathrm{e}^{\mathrm{i}\,\alpha}\right)\left(x - \mathrm{e}^{-\mathrm{i}\,\alpha}\right).$$

Hence,

$$\begin{aligned}
\frac{1}{x^2 - 2\cos(\alpha)x + 1} &= \frac{1}{(x - \mathrm{e}^{\mathrm{i}\,\alpha})(x - \mathrm{e}^{-\mathrm{i}\,\alpha})} \\
&= \frac{1}{(1 - x\,\mathrm{e}^{-\mathrm{i}\,\alpha})(1 - x\,\mathrm{e}^{-\mathrm{i}\,\alpha})} \\
&= \frac{1}{(\mathrm{e}^{\mathrm{i}\,\alpha} - \mathrm{e}^{-\mathrm{i}\,\alpha})x}\left(\frac{1}{1 - x\,\mathrm{e}^{\mathrm{i}\,\alpha}} - \frac{1}{1 - x\,\mathrm{e}^{-\mathrm{i}\,\alpha}}\right) \\
&= \frac{2\,\mathrm{i}}{\sin(\alpha)\,x}\left(\sum_{n=0}^\infty x^n\,\mathrm{e}^{n\,\mathrm{i}\,\alpha} - \sum_{n=0}^\infty x^n\,\mathrm{e}^{-n\,\mathrm{i}\,\alpha}\right) \\
&= \frac{2\,\mathrm{i}}{\sin(\alpha)}\sum_{n=0}^\infty x^{n-1}\left(\mathrm{e}^{n\,\mathrm{i}\,\alpha} - \mathrm{e}^{-n\,\mathrm{i}\,\alpha}\right) \\
&= \frac{2\,\mathrm{i}}{\sin(\alpha)}\sum_{n=0}^\infty x^{n-1}\frac{\sin(n\alpha)}{2\,\mathrm{i}} \\
&= \frac{1}{\sin(\alpha)}\sum_{n=1}^\infty x^{n-1}\sin(n\alpha).
\end{aligned}$$

Therefore,

$$\begin{aligned} I(\alpha) &= \frac{\pi}{2}\frac{1}{\sin(\alpha)}\int_0^1 \sum_{n=1}^{\infty} x^{n-1}\sin(n\alpha)\ln^2(x)\,\mathrm{d}x \\ &= \frac{\pi}{2}\frac{1}{\sin(\alpha)}\sum_{n=1}^{\infty}\sin(n\alpha)\int_0^1 x^{n-1}\ln^2(x)\,\mathrm{d}x \\ &= \frac{\pi}{\sin(\alpha)}\sum_{n=1}^{\infty}\frac{\sin(n\alpha)}{n^3}. \end{aligned}$$

If $\alpha = 0$, then

$$\lim_{\alpha\to 0} I(\alpha) = \pi\sum_{n=1}^{\infty}\lim_{\alpha\to 0}\frac{\sin(n\alpha)}{\sin(\alpha)}\frac{1}{n^3} = \pi\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^3}{6}.$$

Assume $\alpha \neq 0$. Then, by using the Fourier series, we obtain the following

$$\sum_{n=1}^{\infty}\frac{\sin(n\alpha)}{n} = \frac{\pi-\alpha}{2}.$$

Integrate with respect to α, and obtain

$$\begin{aligned} \int_0^{\alpha}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}\,\mathrm{d}x = \sum_{n=1}^{\infty}\int_0^{\alpha}\frac{\sin(nx)}{n}\,\mathrm{d}x &= \sum_{n=1}^{\infty}\frac{1-\cos(\alpha n)}{n^2} \\ = \int_0^{\alpha}\frac{\pi-x}{2}\,\mathrm{d}x &= \frac{\alpha\pi}{2} - \frac{\alpha^2}{4}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^{\alpha}\sum_{n=1}^{\infty}\frac{1-\cos(\alpha n)}{n^2} &= \sum_{n=1}^{\infty}\int_0^{\alpha}\frac{1-\cos(nx)}{n^2}\,\mathrm{d}x = \sum_{n=1}^{\infty}\left(\frac{\alpha}{n^2} - \frac{\sin(n\alpha)}{n^3}\right) \\ &= \int_0^{\alpha}\left(\frac{x\pi}{2} - \frac{x^2}{4}\right)\mathrm{d}x = \frac{\alpha^2\pi}{4} - \frac{\alpha^3}{12} = \frac{\alpha^2(3\pi-\alpha)}{12}. \end{aligned}$$

Finally, isolating the sum,

$$\sum_{n=1}^{\infty}\frac{\sin(ax)}{n^3} = \alpha\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{\alpha^2(\alpha-3\pi)}{12} = \alpha\left(\frac{\pi^2}{6} + \frac{\alpha(\alpha-3\pi)}{12}\right) = \alpha\frac{\alpha^2 - 3\pi\alpha + 2\pi^2}{12},$$

and we conclude

$$I(\alpha) = \frac{\pi\alpha}{\sin(\alpha)}\frac{\alpha^2-3\pi\alpha+2\pi^2}{12} = \frac{\pi}{12}\frac{\alpha(\alpha-\pi)(\alpha-2\pi)}{\sin(\alpha)}.$$