PROBLEM 12243 (AMERICAN MATHEMATICAL MONTHLY)

Proposed by M. L. Glasser (USA).
For $a > 0$ evaluate

$$I_a = \int_0^a \frac{t}{\sinh t\sqrt{1 - \frac{\sinh^2(t)}{\sinh^2(a)}}}\,\mathrm{d}t.$$

Solution proposed by Tommaso Mannelli Mazzoli, TU Wien, 1100 Vienna, Austria:
Let $t = \operatorname{arctanh}(x \tanh a)$. We have that

$$\sinh(t) = \frac{x\tanh a}{\sqrt{1 - x^2\tanh^2(a)}} \quad \text{and} \quad \mathrm{d}t = \frac{\tanh a}{1 - x^2\tanh^2(a)}\,\mathrm{d}x.$$

Then,

$$\begin{aligned}
I_a &= \int_0^1 \frac{\operatorname{arctanh}(x\tanh a)}{\frac{x\tanh a}{\sqrt{1-x^2\tanh^2(a)}}} \cdot \frac{1}{\sqrt{1 - \frac{x^2\tanh^2(a)}{(1-x^2\tanh^2(a))\sinh^2(a)}}} \cdot \frac{\tanh a}{1 - x^2\tanh^2(a)}\,\mathrm{d}x \\
&= \sinh a \int_0^1 \frac{\operatorname{arctanh}(x\tanh a)}{x\sqrt{\sinh^2(a) - x^2\tanh^2(a)\cosh^2(a)}}\,\mathrm{d}x \\
&= \int_0^1 \frac{\operatorname{arctanh}(x\tanh a)}{x\sqrt{1-x^2}}\,\mathrm{d}x \\
&= \int_0^1 \frac{1}{x\sqrt{1-x^2}} \sum_{n=0}^{\infty} \frac{\tanh^{2n+1}(a)x^{2n+1}}{2n+1}\,\mathrm{d}x \\
&= \sum_{n=0}^{\infty} \frac{\tanh^{2n+1}(a)}{2n+1} \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\,\mathrm{d}x,
\end{aligned}$$

where the interchange of summation and integration is justified by the uniform convergence of the Taylor series.
Let $W_n = \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\,\mathrm{d}x$. By substituting $x = \sin\theta$ and integrating by parts, we get

$$\begin{aligned}
W_n &= \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^{2n}(\theta)\,\mathrm{d}\theta = \frac{2n-1}{2n} \cdot W_{n-1} \\
&= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{5\cdot 3}{4\cdot 2} \cdot W_0 \\
&= \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \\
&= \frac{\pi}{2} \cdot \frac{(2n)!}{2^n n!} \cdot \frac{1}{2^n n!} = \frac{\pi}{2} \cdot \frac{1}{4^n}\binom{2n}{n}
\end{aligned}$$

Hence,

$$I_a = \sum_{n=0}^{\infty} \frac{\tanh^{2n+1}(a)}{2n+1} W_n = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\tanh^{2n+1}(a)}{2n+1} \frac{1}{4^n}\binom{2n}{n} = \frac{\pi}{2}\arcsin(\tanh(a)).$$