PROBLEM 12221

Proposed by N. Batir (Turkey). Prove

$$\int_0^1 \frac{\ln(x^6+1)}{x^2+1}\,\mathrm{d}x = \frac{\pi\ln 6}{2} - 3G,$$

where G is Catalan's constant $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$.

Solution proposed by Tommaso Mannelli Mazzoli, TU Wien, 1040 Vienna, Austria.

First, we note that $x^6+1=(x^2+1)(x^4-x^2+1)=(x^2+1)\left((x^2+1)^2-3x^2\right)$. Hence, after substituting $x=\tan(\theta)$, we have

$$\begin{aligned}
\int_0^1 \frac{\ln(x^6+1)}{x^2+1}\,\mathrm{d}x &= \int_0^1 \frac{\ln(x^2+1)+\ln((x^2+1)^2-3x^2)}{x^2+1}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(1-3\sin^2\theta\cos^2\theta)\,\mathrm{d}\theta - 6\int_0^{\frac{\pi}{4}} \ln(\cos\theta)\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(1-\frac{3}{4}\sin^2 t\right)\mathrm{d}t - 6\int_0^{\frac{\pi}{4}} \ln(\cos\theta)\,\mathrm{d}\theta
\end{aligned}$$

Regarding the first integral, define $J(a)$ as follows:

$$J(\alpha) := \int_0^{\frac{\pi}{2}} \ln\left(1-\alpha^2\sin^2 t\right)\mathrm{d}t.$$

Now, for $-1<\alpha<1$, using the substitution $t=\arctan x$, we get

$$\begin{aligned}
J'(\alpha) &= \int_0^{\frac{\pi}{2}} \frac{-2\alpha\sin^2 t}{1-\alpha^2\sin^2 t}\,\mathrm{d}t = \frac{\pi}{\alpha} - \frac{2}{\alpha}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{1-\alpha^2\sin^2 t} \\
&= \frac{\pi}{\alpha} - \frac{2}{\alpha}\int_0^{+\infty} \frac{\mathrm{d}x}{(1-\alpha^2)x^2+1} = \frac{\pi}{\alpha} - \frac{\pi}{\alpha\sqrt{1-a^2}} = \frac{\pi}{\alpha}\left(1-\frac{1}{\sqrt{1-a^2}}\right).
\end{aligned}$$

Thus, since $J(0)=0$, after substituting $t=\sqrt{1-x^2}$, we obtain

$$\begin{aligned}
J(\alpha) &= \int_0^{\alpha} J'(x)\,\mathrm{d}x = \pi\int_0^{\alpha}\left(1-\frac{1}{\sqrt{1-x^2}}\right)\frac{\mathrm{d}x}{x} = \pi\int_{\sqrt{1-\alpha^2}}^1 \frac{t-1}{1-t^2}\,\mathrm{d}t \\
&= -\pi\int_{\sqrt{1-\alpha^2}}^1 \frac{1}{1+t}\,\mathrm{d}t = \pi\ln\left(\frac{1+\sqrt{1-\alpha^2}}{2}\right).
\end{aligned}$$

Therefore, $J\left(\frac{\sqrt{3}}{2}\right)=\pi\ln\frac{3}{4}$. Regarding the second integral, from the Problem 2 in https://ysharificalc.wordpress.com/2018/05/02/catalans-constant-1/, it follows that

$$-6\int_0^{\frac{\pi}{4}} \ln(\cos\theta)\,\mathrm{d}\theta = -6\left(-\frac{\pi}{4}\ln 2+\frac{G}{2}\right) = \frac{\pi\ln 8}{2} - 3G.$$

Finally, putting everything together,

$$\int_0^1 \frac{\ln(x^6+1)}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2}\ln\frac{3}{4} + \frac{\pi\ln 8}{2} - 3G = \frac{\pi\ln 6}{2} - 3G.$$